math 251 - week 13 - Ch 09 - Numrical Methods

LU-Decomposition Method:

↳ divide into two parts.

step.1: Reduce the matrix into Row Echelon form.

step.2: Place the Reciprocal of the Multiplier.

step.3: Place the negation of the Multiplier.

step.4: Form the decomposition $A = LU$.

Ex) Find the LU decomposition of

$$A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

Sol.

$$\begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \qquad \begin{bmatrix} X & 0 & 0 \\ X & X & 0 \\ X & X & X \end{bmatrix}$$

Convert to RREF

$$\xrightarrow{\frac{R_1}{6}} \begin{bmatrix} 1 & 1/{-3} & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \qquad \begin{bmatrix} 6 & 0 & 0 \\ X & X & 0 \\ X & X & X \end{bmatrix}$$

$$\xrightarrow[-3R_1+R_3]{-9R_1+R_2} \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 5 \end{bmatrix} \qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & X & 0 \\ 3 & X & X \end{bmatrix}$$

$$\xrightarrow{\frac{R_2}{2}} \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 8 & 5 \end{bmatrix} \qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & X & X \end{bmatrix}$$

$$\xrightarrow{-8R_2+R_3} \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & X \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \qquad \underbrace{\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}}_{L}$$

upper-traingular matrix

Lower-traingular matrix

$$A = LU = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex| find LU decomposition of

$$A = \begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}$$

Sol.

$$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix} \qquad \begin{bmatrix} X & 0 \\ X & X \end{bmatrix}$$

$$\xrightarrow{\frac{R_1}{2}} \begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 \\ X & X \end{bmatrix}$$

$$\xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 \\ -1 & X \end{bmatrix}$$

$$\xrightarrow{\frac{R_2}{3}} \underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{U} \qquad \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}}_{L}$$

$$A = LU = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

Ex find LU decomposition of

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

sol.

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \qquad \begin{bmatrix} X & 0 \\ X & X \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ X & X \end{bmatrix}$$

$$\xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 2 & X \end{bmatrix}$$

$$\xrightarrow{-R_2} \underbrace{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}_{U} \qquad \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L}$$

$A = LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

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* Singualr Value decomposition:

<u>def.</u> if A is an $n \times n$ matrix and λ_1, λ_2 are the Eigen Values of $A^T A$, then $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$ are called the Singualr value of A.

<u>Ex</u> find the Singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

<u>Sol.</u>

$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$A^T . A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

the Eigen Values of $A^T A$

the characteristic equation is

$$\Rightarrow |\lambda I - A^T A| = 0$$

$$= \begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix} = 0$$

$$(\lambda-2)(\lambda-2) - (-1)(-1) = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-3)(\lambda-1) = 0$$

$\lambda_1 = 3$, $\lambda_2 = 1$ are the eigenvalues.

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$$

are the singular values.

Ex] find the Singualr value of the matrix

A, if given the following

$$A^T A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Sol. the characteristic equation is:

$$|\lambda I - A^T A| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-4 & 0 & -1 \\ 2 & \lambda-1 & 0 \\ 2 & 0 & \lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$\lambda = 1, 2, 3$ are the Eigenvalues.

the singualr value of A is:

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{1} = 1$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$$

$$\sigma_3 = \sqrt{\lambda_3} = \sqrt{3}$$

sigma

Ex] find the distinct singualr value

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

Sol. $A^T = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

the characteristic equation is:

$$|\lambda I - A^T A| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-5 & 0 \\ 0 & \lambda-5 \end{vmatrix} = 0 \Rightarrow (\lambda-5)(\lambda-5) = 0$$

$\underline{\lambda = 5, 5}$ are the $\underline{\text{E.values}}$

$$\left.\begin{aligned} \sigma_1 &= \sqrt{\lambda_1} = \sqrt{5} \\ \sigma_2 &= \sqrt{\lambda_2} = \sqrt{5} \end{aligned}\right\} \rightarrow \text{are the Singualr Values.}$$

* The Method of LU-decomposition if we have the System of Equations:

Step.1: Rewrite the System $Ax=b$

$$L \quad U$$

or we can write $LUx=b$

Step.2: Let $y=Ux$, then we can write

$$Ly=b$$

Now we can solve this, find the value of y_1, y_2.

Step.3: Now take the value of y from Step.2 and solve the equations

$$y=Ux \quad \text{for } x$$

that will give the Solution $Ax=b$.

Ex] use the method LU decomposition to Solve the System

$$2x - 4y = -2$$
$$-x + 3y = 2$$

Sol.

$$A = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}, x = \begin{bmatrix} x \\ y \end{bmatrix}, b = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix} \quad \begin{bmatrix} X & 0 \\ X & X \end{bmatrix}$$

$$\xrightarrow{\frac{R_1}{2}} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ X & X \end{bmatrix}$$

$$\xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ -1 & X \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}_{U} \quad \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}}_{L}$$

step.2: $Ly = b$

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

we got $y_1 = -1$, $y_2 = 1$ $\Rightarrow$ $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

step.3:

$Ux = y$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now, we get $x = 1$, $y = 1$